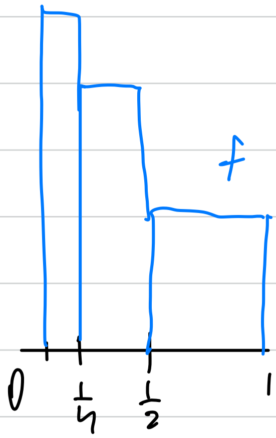
Math 564: Advance Analysis 1
Lecture 14



Integrable functions may not be bounded.

Example. $f := \sum_{n=0}^{\infty} \mathbb{1}_{[2^{-(n+1)}, 2^{-n})} \cdot (1.5)^n$, then $\int f \, d\lambda = \sum 2^{-(n+1)} \cdot (1.5)^n < \infty$.

99% boundedness of L^1 functions. Let (X, μ) be a measure space and $f \in L^1(X, \mu)$. For each $\varepsilon > 0$ there is a μ-meas. $X' \subseteq X$ s.t. $\int |f| \, d\mu \approx_\varepsilon \int_{X'} |f| \, d\mu$, i.e., $\mu_{|f|}(X \setminus X') \leq \varepsilon$, and $f|_{X'}$ is bdd.

Proof. Because $|f| < \infty$ on a conull set, we may assume $|f| < \infty$. We have that $\forall x \, \exists n \; |f| \leq n$. We would like switch these quantifiers, paying an ε price. Let $X_n := \{x \in X : |f(x)| \leq n\}$, then $X_n \nearrow X$ so, $\mu_{|f|}(X_n) \nearrow \mu_{|f|}(X)$.
Because $\mu_{|f|}(X) = \|f\|_1 < \infty$, we get $\mu_{|f|}(X \setminus X_n) \leq \varepsilon$ for large enough n. By Chebyshev, we also have $\mu(X \setminus X_n) \leq \frac{1}{n} \cdot \|f\|_1$, hence for large enough n, $\mu(X \setminus X_n) < \varepsilon$. $\square$

Def. For measures μ, ν on a measurable space $(X, \mathcal{B})$, we say that ν is absolutely continuous with respect to μ, and write $\nu \ll \mu$, if μ-null $\Rightarrow$ ν-null, i.e., if a set $B \in \mathcal{B}$ is μ-null, then it's also ν-null.

Example. For any $f \in L^+(X, \mu)$, we have $\mu_f \ll \mu$.

The name "absolute continuity" comes from an equiv. def. to $\nu \ll \mu$ finite measure ν.

Prop. Let ν, μ be measures on a measurable space $(X, \mathcal{B})$, where ν is finite. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for each $B \in \mathcal{B}$, we have $\mu(B) < \delta \Rightarrow \nu(B) < \varepsilon$.

Proof. $\Leftarrow$. Trivial.

$\Rightarrow$. Given $\varepsilon > 0$, suppose towards a contradiction that $\forall \delta > 0$ $\exists B_\delta \in \mathcal{B}$ s.t. $\mu(B_\delta) < \delta$ but $\nu(B_\delta) \geq \varepsilon$. By Borel-Cantelli (recall the first application), there a decreasing sequence (B_n) s.t. $\bigcap_n B_n$ is μ-null, yet $\nu(B_n) \geq \varepsilon$. But μ-null $\Rightarrow$ ν-null, so $\bigcap_n B_n$ is also ν-null, contradicting $\nu(B_n) \searrow \nu(\bigcap_n B_n) = 0$. $\square$

Cor. If $f \in L^1$, then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\mu(B) < \delta \Rightarrow \int_B |f| \, d\mu < \varepsilon$, for all μ-measurable sets B.

Proof. $\mu_{|f|}(X) = \|f\|_1 < \infty$, so $\mu_{|f|} \ll \mu$ implies the conclusion. $\square$

Convergence in measure. We saw that $f_n \to f$ ptwise doesn't always imply L^1-convergence. Conversely, it is also false that L^1-convergence implies ptwise convergence. Let's at least show that $f_n \to_{L^1} f$ implies that $f_{n_k} \to f$ a.e. for some subsequence. To show this, we will introduce the third notion of convergence and prove via this intermediate step.

Examples (Folland).

i. $f_n = n^{-1}\chi_{(0,n)}$.

ii. $f_n = \chi_{(n,n+1)}$.

iii. $f_n = n\chi_{[0,1/n]}$.

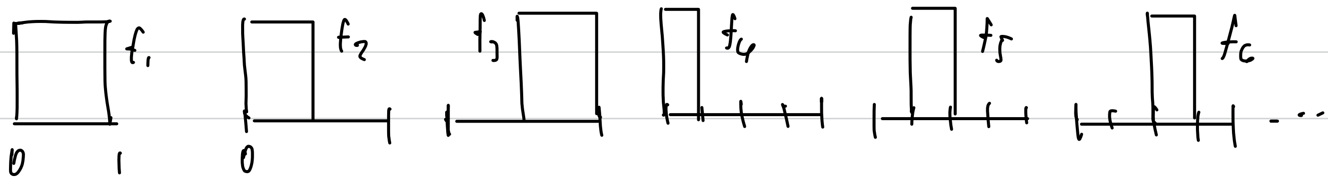
iv. $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,1/2]}$, $f_6 = \chi_{[1/2,3/4]}$, $f_7 = \chi_{[3/4,1]}$, and in general, $f_n = \chi_{[j/2^k,\,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$.

(i) $f_n \to 0$ ptwise, in fact uniformly, but $\int f_n \, dx = 1 \not\to 0$, so not in L^1.

(ii) $f_n \to 0$ ptwise, but not uniformly and not in L^1.

(iii) $f_n \to 0$ a.e., but not uniformly and not in L^1.

(iv)



$\int f_n \, d\lambda \to 0$ hence in L^1, but $f_n \not\to f$ a.e. because for every x, $(f_n(x))$ has ∞-many 0 and ∞-many 1. But $f_{2^k} \to 0$ a.e.

Let (X, μ) be a measure space.

Def. For $\alpha \geq 0$ and μ-meas. real-valued functions f, g, define

$$\Delta_\alpha(f,g) := \{x \in X : |f(x) - g(x)| \geq \alpha\},$$

$$\delta_\alpha(f,g) := \mu(\Delta_\alpha(f,g)).$$

These δ_α are not even pseudo-metrics; indeed, let $f \equiv 0$, $g \equiv 1$, $h \equiv 2$, then $\delta_2(f,g) = \delta_2(g,h) = 0$ but $\delta_2(f,h) = \mu(X)$. However this family satisfies:

Quasi-Δ-ineq. (a) $\Delta_{\alpha+\beta}(f,h) \subseteq \Delta_\alpha(f,g) \cup \Delta_\beta(g,h)$

(b) $\delta_{\alpha+\beta}(f,h) \leq \delta_\alpha(f,g) + \delta_\beta(g,h)$.

Proof. If $|f(x) - h(x)| \geq \alpha + \beta$ then either $|f(x) - g(x)| \geq \alpha$ or $|g(x) - h(x)| \geq \beta$, by the Δ-ineq. for reals. $\square$

Def. ∘ We say that a sequence (f_n) *converges in measure* to f, and write $f_n \to_\mu f$, if $\forall \alpha > 0$, $\delta_\alpha(f_n, f) = \mu(\{x \in X : |f_n(x) - f_n(x)| \geq \alpha\}) \to 0$.

∘ We say that (f_n) is *Cauchy in measure* if $\forall \alpha > 0$,
$$\delta_\alpha(f_n, f_m) \to 0 \quad \text{as } n, m \to \infty.$$

Examples (Folland). (i) $f_n \to_\mu 0$

(ii) $f_n \not\to_\mu 0$

(iii) $f_n \to_\mu 0$

(iv) $f_n \to_\mu 0$

L_1-conv. $\Rightarrow$ meas-conv. If $f_n \to_{L_1} f$ then $f_n \to_\mu f$.

Proof. Fix $\alpha > 0$. Then by Chebyshev, $\delta_\alpha(f_n, f) \leq \frac{1}{\alpha} \cdot \|f - f_n\|_1 \to 0$. □

Measure-convergence top is Hausdorff mod null. If $f_n \to_\mu f$ and $f_n \to_\mu g$, then $f = g$ a.e.

Proof. Fix $\alpha > 0$. It's enough to show $\delta_\alpha(f, g) = 0$. By quasi-Δ-ineq:
$$\delta_\alpha(f, g) \leq \delta_{\alpha/2}(f, f_n) + \delta_{\alpha/2}(f_n, g) \to 0, \quad \text{as } n \to \infty.$$ □

Prop. If (f_n) is Cauchy in measure and $(f_{n_k}) \to_\mu f$ for some subsequence, then $f_n \to_\mu f$ in measure.

Proof. Again by quasi-Δ-ineq., HW.

Thm. If $f_n \to_\mu f$ in measure (e.g. when $f_n \to_{L^1} f$), there is a subsequence (f_{n_k}) that converges to f a.e.